Eur. Phys. J. B **50**, 445–452 (2006)
DOI: 10.1140/epjb/e2006-00156-3

THE EUROPEAN PHYSICAL JOURNAL B

# Some quasi-periodic solutions to the Kadometsev-Petviashvili and modified Kadometsev-Petviashvili equations

J. Chen[a] and X. Geng

Department of Mathematics, Zhengzhou University Zhengzhou 450052, Henan, P.R. China

Received 22 September 2005 / Received in final form 6 December 2005
Published online 5 May 2006 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2006

**Abstract.** The Kadometsev-Petviashvili (KP) and modified KP (mKP) equations are retrieved from the first two soliton equations of coupled Korteweg-de Vries (cKdV) hierarchy. Based on the nonlinearization of Lax pairs, the KP and mKP equations are ultimately reduced to integrable finite-dimensional Hamiltonian systems in view of the r-matrix theory. Finally, the resulting Hamiltonian flows are linearized in Abel-Jacobi coordinates, such that some specially explicit quasi-periodic solutions to the KP and mKP equations are synchronously given in terms of theta functions through the Jacobi inversion.

**PACS.** 02.30.IK Integrable systems – 02.30.Jr Partial differential equations

## 1 Introduction

Integrable models play a prominent role in theoretical physics. The reason is not only the direct phenomenological interest of some of them, but also the fact that they often provide some deep insights into the mathematical structure of theory in which they arise. Up to now, the (1+1)-dimensional integrable models are well understood due to many systematic methods such as the inverse scattering transformation, the dressing method, the Darboux transformation and the algebro-geometric method [1–7]. However, studies of the (2+1)-dimensional cases are fewer in number and such systems are being actively investigated from different viewpoints. Recently, it is worthwhile to mention that the nonlinearization of Lax pairs [8,9] has been generalized to study multi-dimensional nonlinear evolution equations consisting of three steps: decomposition, linearization and inversion of the flows [10–15]. The most important message relating to the progress of this manipulation is that compatible solutions of soliton equations naturally give rise to the solutions of (2+1)-dimensional nonlinear evolution equations [16,17].

In this paper the well-known KP [18] and mKP [19] equations are simultaneously studied from a different decomposition and generalized treatment. The main objective is to display their underlying linear behavior on a Riemann surface of hyperelliptic curve, and further give some new quasi-periodic solutions exhibiting the characteristic of Liouville integrability. This current paper extends the above-mentioned fruitful method [10–15] deriving quasi-periodic solutions for multi-dimensional nonlinear evolution equations. Here the techniques involved are the theory of algebraic curves, the r-matrix theory and the nonlinearization of Lax pairs. In particular, it is in this constructive framework that the KP and mKP equations can be settled simultaneously with the help of the cKdV soliton hierarchy. Details of the organization of the present paper are as follows. In Section 2, the KP and mKP equations are recovered from the first two nontrivial members in the cKdV soliton hierarchy. This implies that (2+1)-dimensional systems are decomposed into (1+1)-dimensional systems that are easier to treat with some available tools. Section 3 further reduces the corresponding (1+1)-dimensional systems into integrable finite-dimensional Hamiltonian systems (FDHSs) by means of the Bargmann constraint. In Section 4, Abel-Jacobi coordinates are introduced to straighten out the resulting Hamiltonian flows, indicating their underlying linearities in the form of Abel-Jacobi variables. The Jacobi inversion for writing out theta function solutions to the KP and mKP equations is the subject of Section 5.

## 2 An alternative construction of the KP and mKP equations

In this section, we briefly recollect some necessary formulae and deduce the KP and mKP equations. We start with the cKdV spectral problem [20,21],

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u & -v \\ 1 & \frac{1}{2}\lambda - \frac{1}{2}u \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{2.1}$$

[a] e-mail: jbchen@gs.zzu.edu.cn

where $\lambda$ is a constant spectral parameter; and $u, v$ are two potentials. Consider the stationary zero-curvature equation of (2.1),

$$V_x = [U, V], \quad V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{j\geq 0} \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \lambda^{-j}, \tag{2.2}$$

which are in agreement with

$$\begin{gathered} a_j = -\partial^{-1}(vc_j + b_j), \\ \begin{pmatrix} b_{j+1} \\ c_{j+1} \end{pmatrix} = \begin{pmatrix} -\partial + u - 2v\partial^{-1} & -2v\partial^{-1}v \\ 2\partial^{-1} & \partial + u + 2\partial^{-1}v \end{pmatrix} \begin{pmatrix} b_j \\ c_j \end{pmatrix}, \end{gathered} \tag{2.3}$$

where $\partial = \partial/\partial x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$. Designating initial values by

$$a_0 = \frac{1}{2}, \qquad b_0 = c_0 = 0,$$

which together with (2.3) yields

$$\begin{aligned} & a_1 = 0, \quad b_1 = v, \quad c_1 = -1, \\ & a_2 = v, \quad b_2 = -v_x + uv, \quad c_2 = -u, \\ & a_3 = 2uv - v_x, \quad b_3 = v_{xx} - u_x v \\ & \qquad - 2uv_x + u^2 v + 2v^2, \quad c_3 = -u_x - u^2 - 2v, \\ & a_4 = v_{xx} + 3v^2 - 3uv_x + 3u^2 v, \\ & b_4 = -v_{xxx} + u_{xx} v + 3u_x v_x + 3uv_{xx} \\ & \qquad - 3u^2 v_x - 3uu_x v + 6uv^2 - 6vv_x + u^3 v, \\ & c_4 = -u_{xx} - 3uu_x - u^3 - 6uv. \end{aligned} \tag{2.4}$$

To deduce the cKdV soliton hierarchy, let us introduce an auxiliary spectral problem of (2.1),

$$\varphi_{t_n} = V^{(n)}\varphi, \quad V^{(n)} = \begin{pmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & -V_{11}^{(n)} \end{pmatrix}, \qquad n \geq 1, \tag{2.5}$$

where

$$\begin{aligned} V_{11}^{(n)} &= \frac{1}{2}c_{n+1} + \sum_{j=0}^{n} a_j \lambda^{n-j}, \\ V_{12}^{(n)} &= \sum_{j=0}^{n} b_j \lambda^{n-j}, \\ V_{21}^{(n)} &= \sum_{j=0}^{n} c_j \lambda^{n-j}. \end{aligned}$$

The compatibility condition of (2.1) and (2.5), i.e. $\varphi_{xt_n} = \varphi_{t_n x}$, leads to the zero-curvature equation

$$U_{t_n} - V_x^{(n)} + \left[U, V^{(n)}\right] = 0, \tag{2.6}$$

which is the cKdV soliton hierarchy

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ -c_{n+1} \end{pmatrix} \triangleq Jg_{n-1}, \qquad n \geq 1. \tag{2.7}$$

After a direct calculation, it is easy to list the cKdV equation and the next one with $y = t_2$ and $t = t_3$ respectively,

$$\begin{cases} u_y = -u_{xx} - 2uu_x - 2v_x, \\ v_y = v_{xx} - 2u_x v - 2uv_x, \end{cases} \tag{2.8}$$

$$\begin{cases} u_t = -u_{xxx} - 3(uu_x + 2uv)_x - 3u^2 u_x, \\ v_t = -v_{xxx} + 3(uv_x - u^2 v)_x - 6vv_x. \end{cases} \tag{2.9}$$

Actually the cKdV equation (2.8) is the compatibility condition of (2.1) and

$$\begin{aligned} & \varphi_y = V^{(2)}\varphi, \\ & V^{(2)} = \begin{pmatrix} \frac{1}{2}\lambda^2 - \frac{1}{2}u_x - \frac{1}{2}u^2 & \lambda v - v_x + uv \\ -\lambda - u & -\frac{1}{2}\lambda^2 + \frac{1}{2}u_x + \frac{1}{2}u^2 \end{pmatrix}, \end{aligned} \tag{2.10}$$

while (2.9) is the compatibility condition of (2.1) and

$$\varphi_t = V^{(3)}\varphi, \qquad V^{(3)} = \begin{pmatrix} V_{11}^{(3)} & V_{12}^{(3)} \\ V_{21}^{(3)} & -V_{11}^{(3)} \end{pmatrix}, \tag{2.11}$$

with

$$\begin{aligned} V_{11}^{(3)} &= \frac{1}{2}\lambda^3 + \lambda v - \frac{1}{2}u_{xx} - \frac{3}{2}uu_x - \frac{1}{2}u^3 - uv - v_x, \\ V_{12}^{(3)} &= \lambda^2 v - \lambda v_x + \lambda uv + v_{xx} - u_x v - 2uv_x + u^2 v + 2v^2, \\ V_{21}^{(3)} &= -\lambda^2 - \lambda u - u_x - u^2 - 2v. \end{aligned}$$

Let $u, v$ be the compatible solutions of (2.8) and (2.9), and

$$g = u(x, y, t), \qquad h = v(x, y, t). \tag{2.12}$$

From (2.8) and (2.12), a direct calculation delivers

$$\begin{aligned} \partial_x^{-1} g_y &= -g_x - g^2 - 2h, \\ \partial_x^{-1} g_{yy} &= g_{xxx} + 2g_x^2 + 4gg_{xx} + 4g^2 g_x + 8gh_x + 4g_x h, \\ \partial_x^{-1} h_y &= h_x - 2gh, \\ \partial_x^{-1} h_{yy} &= h_{xxx} - 4g_x h_x - 4gh_{xx} + 8gg_x h + 4hh_x + 4g^2 h_x. \end{aligned} \tag{2.13}$$

Combining (2.13) and (2.9) gives the following (2+1)-dimensional nonlinear evolution equations,

$$g_t = -\frac{1}{4}(g_{xx} - 2g^3)_x - \frac{3}{4}(\partial_x^{-1} g_{yy} - 2g_x \partial_x^{-1} g_y), \tag{2.14}$$

$$h_t = -\frac{1}{4}h_{xxx} - 3hh_x - \frac{3}{4}\partial_x^{-1} h_{yy}. \tag{2.15}$$

It is remarkable to see that (2.14) is the mKP equation and (2.15) is the KP equation, which are closely related to soliton equations (2.8) and (2.9). Noting the compatibility of (2.8) and (2.9) in the same soliton hierarchy (we will return later to the compatibility), it suggests that (2.12) solves (2.14) and (2.15) simultaneously.

**Remark:** Though the coefficients of (2.14) and (2.15) are different from their presentations in the literatures [18,19]; in fact, they are identical to each other in view of a simple linear transformation with regard to the scalars $x$, $y$ and $t$.

## 3 The associated Hamiltonian systems

This section applies the nonlinearization of Lax pairs to the cKdV spectral problem and its auxiliary spectral problem such that the resulting (2+1)-dimensional systems are ultimately reduced into three integrable FDHSs. Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be $N$ distinct eigenvalues and $\varphi = (p_j, q_j)^T$ be the eigenfunction. We take $N$ copies of the spectral problem (2.1),

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2}\lambda_j + \frac{1}{2}u & -v \\ 1 & \frac{1}{2}\lambda_j - \frac{1}{2}u \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix}, \quad 1 \le j \le N. \tag{3.1}$$

A simple computation provides [21]

$$\nabla\lambda_j = (\delta\lambda_j/\delta u, \delta\lambda_j/\delta v)^T = \left(-p_j q_j, q_j^2\right)^T. \tag{3.2}$$

Taking into consideration the Bargmann constraint [8,9],

$$g_0 = \sum_{j=1}^{N} \nabla\lambda_j, \tag{3.3}$$

which gives rise to

$$u = \langle q, q\rangle, \qquad v = -\langle p, q\rangle, \tag{3.4}$$

where $p = (p_1, \cdots, p_N)^T, q = (q_1, \cdots, q_N)^T$, and $\langle\cdot,\cdot\rangle$ stands for the standard inner product in $\mathbb{R}^N$. According to the principle of the nonlinearization of Lax pairs, submitting (3.4) into (3.1) gives the following FDHS

$$\begin{cases} p_x = -\frac{1}{2}\Lambda p + \frac{1}{2}\langle q, q\rangle p + \langle p, q\rangle q = -\frac{\partial H_0}{\partial q}, \\ q_x = p + \frac{1}{2}\Lambda q - \frac{1}{2}\langle q, q\rangle q = \frac{\partial H_0}{\partial p}, \end{cases} \tag{3.5}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, and

$$H_0 = \frac{1}{2}(\langle p, p\rangle + \langle\Lambda p, q\rangle - \langle p, q\rangle\langle q, q\rangle). \tag{3.6}$$

Analogously we nonlinearize the temporal parts of the soliton hierarchy. Utilizing (2.10), (2.11), (3.4) and (3.5), it is not difficult to obtain, respectively,

$$\begin{cases} p_y = \frac{1}{2}\Lambda^2 p - \langle p, q\rangle p - \frac{1}{2}\langle\Lambda q, q\rangle p \\ \qquad -\langle p, q\rangle\Lambda q + \langle p, p\rangle q = -\frac{\partial H_1}{\partial q}, \\ q_y = -\Lambda p - \langle q, q\rangle p - \frac{1}{2}\Lambda^2 q + \langle p, q\rangle q \\ \qquad +\frac{1}{2}\langle\Lambda q, q\rangle q = \frac{\partial H_1}{\partial p}, \end{cases} \tag{3.7}$$

with

$$H_1 = -\frac{1}{2}\left(\langle\Lambda^2 p, q\rangle - \langle p, q\rangle^2 - \langle\Lambda q, q\rangle\langle p, q\rangle + \langle p, p\rangle\langle q, q\rangle + \langle\Lambda p, p\rangle\right), \tag{3.8}$$

and

$$\begin{cases} p_t = \frac{1}{2}\Lambda^3 p - \langle p, q\rangle\Lambda p - \langle\Lambda p, q\rangle p - \frac{1}{2}\langle\Lambda^2 q, q\rangle p - \langle p, q\rangle\Lambda^2 q \\ \qquad +\langle p, p\rangle\Lambda q + \langle\Lambda p, p\rangle q = -\frac{\partial H_2}{\partial q}, \\ q_t = -\Lambda^2 p - \langle q, q\rangle\Lambda p - \langle\Lambda q, q\rangle p - \frac{1}{2}\Lambda^3 q + \langle p, q\rangle\Lambda q \\ \qquad +\langle\Lambda p, q\rangle q + \frac{1}{2}\langle\Lambda^2 q, q\rangle q = \frac{\partial H_2}{\partial p}, \end{cases} \tag{3.9}$$

with

$$H_2 = -\tfrac{1}{2}(\langle\Lambda^3 p, q\rangle - \langle\Lambda^2 q, q\rangle\langle p, q\rangle + \langle p, p\rangle\langle\Lambda q, q\rangle + \langle\Lambda p, p\rangle\langle q, q\rangle + \langle\Lambda^2 p, p\rangle) + \langle p, q\rangle\langle\Lambda p, q\rangle. \tag{3.10}$$

Summing up these discussions, a direct calculation implying that (2.14) and (2.15) are satisfied by (3.4) with the help of (3.5), (3.7) and (3.9) is formulated as below.

**Proposition 1** *Let $(p(x,y,t), q(x,y,t))^T$ be the compatible solution of $H_0$, $H_1$ and $H_2$. Then $g(x,y,t) = \langle q, q\rangle$ solves the mKP equation; $h(x,y,t) = -\langle p, q\rangle$ solves the KP equation.*

In what follows, we are in a position to show the Liouville integrability of the resulting FDHSs. In a similar way to the treatment of [22], a lengthy calculation gives the following Lax equations, which play an important role in our argument.

**Proposition 2** *The FDHSs (3.5), (3.7) and (3.9) admitting a Lax representation $L(\lambda)$ enjoy Lax equations, respectively,*

$$\begin{aligned} L_x(\lambda) &= [\bar{U}, L(\lambda)], \\ L_y(\lambda) &= [\bar{V}^{(2)}, L(\lambda)], \\ L_t(\lambda) &= [\bar{V}^{(3)}, L(\lambda)], \end{aligned} \tag{3.11}$$

*where*

$$L(\lambda) = \begin{pmatrix} \frac{1}{2}\lambda & -\langle p, q\rangle \\ -1 & -\frac{1}{2}\lambda \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} -p_j q_j & p_j^2 \\ -q_j^2 & p_j q_j \end{pmatrix} \triangleq \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \tag{3.12}$$

$$\bar{U} = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}\langle q, q\rangle & \langle p, q\rangle \\ 1 & \frac{1}{2}\lambda - \frac{1}{2}\langle q, q\rangle \end{pmatrix}, \tag{3.13}$$

$$\bar{V}^{(2)} = \begin{pmatrix} \frac{1}{2}\lambda^2 - \langle p, q\rangle - \frac{1}{2}\langle\Lambda q, q\rangle & -\lambda\langle p, q\rangle + \langle p, p\rangle \\ -\lambda - \langle q, q\rangle & -\frac{1}{2}\lambda^2 + \langle p, q\rangle + \frac{1}{2}\langle\Lambda q, q\rangle \end{pmatrix}, \tag{3.14}$$

*and*

$$\bar{V}^{(3)} = \begin{pmatrix} \bar{V}_{11}^{(3)} & -\lambda^2\langle p,q\rangle + \lambda\langle p,p\rangle + \langle \Lambda p,p\rangle \\ -\lambda^2 - \lambda\langle q,q\rangle - \langle \Lambda q,q\rangle & -\bar{V}_{11}^{(3)} \end{pmatrix}, \tag{3.15}$$

*with*

$$\bar{V}_{11}^{(3)} = \frac{1}{2}\lambda^3 - \lambda\langle p,q\rangle - \langle \Lambda p,q\rangle - \frac{1}{2}\langle \Lambda^2 q,q\rangle.$$

To prove the Liouville integrability, we first recall some fundamental concepts. The Poisson bracket of two functions $F$ and $G$ in the symplectic space $(\mathbb{R}^{2N}, dp\wedge dq)$ is defined as [25]

$$\{F,G\} = \sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right) = \left\langle \frac{\partial F}{\partial q}, \frac{\partial G}{\partial p}\right\rangle - \left\langle \frac{\partial F}{\partial p}, \frac{\partial G}{\partial q}\right\rangle.$$

An immediate consequence is that (3.5), (3.7) and (3.9) can be rewritten as, respectively,

$$p_x = \{p, H_0\}, \qquad q_x = \{q, H_0\},$$
$$p_y = \{p, H_1\}, \qquad q_y = \{q, H_1\},$$
$$p_t = \{p, H_2\}, \qquad q_t = \{q, H_2\}.$$

Denoting the standard notation [23] by

$$L_1(\lambda) = L(\lambda)\otimes I, \qquad L_2(\mu) = I\otimes L(\mu),$$

where $I$ is the $2\times 2$ unit matrix, and

$$\{L_1(\lambda), L_2(\mu)\}^{jk,mn} = \left\{L_1(\lambda)^{jm}, L_2(\mu)^{kn}\right\}.$$

Therefore, in the standard symplectic structure $(\mathbb{R}^{2N}, dp\wedge dq)$, it is not difficult to check that $L(\lambda)$ satisfies a classical r-matrix formula

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda,\mu), L_1(\lambda)] + [r_{21}(\lambda,\mu), L_2(\mu)], \tag{3.16}$$

where

$$r_{12} = \frac{2}{\mu-\lambda}P + Q_{12}, \qquad r_{21} = \frac{2}{\mu-\lambda}P + Q_{21}, \tag{3.17}$$

with

$$P = \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix},$$

$$Q_{12} = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&0&0 \end{pmatrix}, Q_{21} = \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}.$$

From [24], it is immediately shown that

$$\{\det L(\lambda), \det L(\mu)\} = 0. \tag{3.18}$$

On the other hand, $\det L(\lambda)$ can be rewritten as

$$\det L(\lambda) = -\frac{1}{4}\lambda^2 + \sum_{j=1}^{N}\frac{E_j}{\lambda-\lambda_j}, \tag{3.19}$$

where

$$E_j = p_j^2 + \lambda_j p_j q_j - \langle p,q\rangle q_j^2 + \sum_{i=1,i\neq j}^{N}\frac{(p_j q_i - p_i q_j)^2}{\lambda_j - \lambda_i}, \quad j = 1,2,\cdots,N.$$

From (3.18), it is clear that

$$\{E_j, E_k\} = 0, \qquad 1\le j,k\le N.$$

Simply taking $F_k = \sum_{j=1}^{N}\lambda_j^k E_j$, it is known that all Hamiltonian systems $(\mathbb{R}^{2N}, dp\wedge dq, F_k)$ are completely integrable in the Liouville sense. Noting that

$$H_0 = \frac{1}{2}F_0, \qquad H_1 = -\frac{1}{2}F_1, \qquad H_2 = -\frac{1}{2}F_2,$$

we conclude the following theorem.

**Theorem 1** *The FDHSs (3.5), (3.7) and (3.9) are completely integrable in the Liouville sense.*

**Note.** Due to the involutivity of $H_0$, $H_1$ and $H_2$, the corresponding flows mutually commute [25]; this grants the compatibility of (3.5), (3.7) and (3.9).

## 4 Straightening out the Hamiltonian flows

Firstly we introduce two sets of elliptic coordinates $\mu_1,\mu_2,\cdots,\mu_N$ and $\nu_1,\nu_2,\cdots,\nu_N$ for the integrable FDHSs (3.5), (3.7) and (3.9) from $L(\lambda)$ [26,27],

$$\begin{aligned} B(\lambda) &= -\langle p,q\rangle + \sum_{j=1}^{N}\frac{p_j^2}{\lambda-\lambda_j} = -\langle p,q\rangle\frac{m(\lambda)}{a(\lambda)}, \\ C(\lambda) &= -1 - \sum_{j=1}^{N}\frac{q_j^2}{\lambda-\lambda_j} = -\frac{n(\lambda)}{a(\lambda)}, \end{aligned} \tag{4.1}$$

where

$$a(\lambda) = \prod_{k=1}^{N}(\lambda-\lambda_k),$$
$$m(\lambda) = \prod_{k=1}^{N}(\lambda-\mu_k),$$
$$n(\lambda) = \prod_{k=1}^{N}(\lambda-\nu_k). \tag{4.2}$$

Resorting to (4.1) and (4.2), it is easy to calculate that

$$\frac{\langle p,p\rangle}{\langle p,q\rangle}=\sigma_1-\sigma,\qquad \langle q,q\rangle=\sigma-\sigma_2,\tag{4.3}$$

where

$$\sigma=\sum_{j=1}^{N}\lambda_j,\qquad \sigma_1=\sum_{j=1}^{N}\mu_j,\qquad \sigma_2=\sum_{j=1}^{N}\nu_j.$$

Noting (2.12) and (3.4), a simple calculation delivers

$$g=-\sigma_2+\sigma,\qquad \partial\ln h=\sigma_1-\sigma_2.\tag{4.4}$$

The combination of (4.1) and (4.2) also implies that

$$\bar V_{12}^{(2)}=v(\lambda-\sigma_1+\sigma),\qquad \bar V_{21}^{(2)}=-\lambda+\sigma_2-\sigma,\tag{4.5}$$

$$\begin{cases}\bar V_{12}^{(3)}&=v(\lambda^2-(\lambda+\sigma)(\sigma_1-\sigma)-\bar\sigma+\bar\sigma_1),\\ \bar V_{21}^{(3)}&=-\lambda^2+(\lambda+\sigma)(\sigma_2-\sigma)-\bar\sigma_2+\bar\sigma,\end{cases}\tag{4.6}$$

where

$$\bar\sigma=\sum_{i,j=1,i\langle j}^{N}\lambda_i\lambda_j,\quad \bar\sigma_1=\sum_{i,j=1,i\langle j}^{N}\mu_i\mu_j,\quad \bar\sigma_2=\sum_{i,j=1,i\langle j}^{N}\nu_i\nu_j.\tag{4.7}$$

To go on, let us define

$$\det L(\lambda)=-A(\lambda)^2-B(\lambda)C(\lambda)=-\frac{b(\lambda)}{4a(\lambda)}=-\frac{R(\lambda)}{4a^2(\lambda)},\tag{4.8}$$

where

$$b(\lambda)=\prod_{k=1}^{N+2}(\lambda-\lambda_{N+k}),$$
$$R(\lambda)=a(\lambda)b(\lambda)=\prod_{k=1}^{2N+2}(\lambda-\lambda_k).$$

Recalling (4.1), we have

$$A(\mu_k)=\frac{\sqrt{R(\mu_k)}}{2a(\mu_k)},\qquad A(\nu_k)=\frac{\sqrt{R(\nu_k)}}{2a(\nu_k)},\qquad 1\le k\le N.\tag{4.9}$$

From $(4.1)_1$, it is apparent that

$$\left.\frac{dB}{dx}\right|_{\mu_k}=\langle p,q\rangle\frac{\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)}{a(\mu_k)}\frac{d\mu_k}{dx}.\tag{4.10}$$

Appealing to the Lax equation $(3.11)_1$, we derive the evolution equation of elliptic coordinate $\mu_k$ with respect to the variable $x$

$$\frac{d\mu_k}{dx}=-\frac{\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)},\qquad 1\le k\le N.\tag{4.11}$$

In a similar way, we arrive at

$$\frac{d\nu_k}{dx}=\frac{\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i)},\qquad 1\le k\le N.\tag{4.12}$$

Analogously we achieve the evolution equations of all $\mu_k$ and $\nu_k$ in variables $y$ and $t$,

$$\begin{cases}\frac{d\mu_k}{dy}&=\frac{(\mu_k-\sigma_1+\sigma)\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)},\\ \frac{d\nu_k}{dy}&=\frac{(-\nu_k+\sigma_2-\sigma)\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i)},\end{cases}\qquad 1\le k\le N,\tag{4.13}$$

and

$$\begin{cases}\frac{d\mu_k}{dt}&=\frac{\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)}\Big(\mu_k^2-(\mu_k+\sigma)\\ &\qquad\times(\sigma_1-\sigma)-\bar\sigma+\bar\sigma_1\Big),\\ \frac{d\nu_k}{dt}&=\frac{\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i)}\Big(-\nu_k^2+(\nu_k+\sigma)\\ &\qquad\times(\sigma_2-\sigma)-\bar\sigma_2+\bar\sigma\Big),\end{cases}\qquad 1\le k\le N.\tag{4.14}$$

Next we are ready to linearize the Hamiltonian flows. For this purpose let us introduce the Riemann surface $\Gamma$ of hyperelliptic curve given by the affine equation $\xi^2=R(\lambda)$, which is the genus of $N$. For the same $\lambda$, there exist two points $(\lambda,\sqrt{R(\lambda)})$ and $(\lambda,-\sqrt{R(\lambda)})$ on the upper and lower sheets of $\Gamma$. In addition, there are two infinite points that are not the branch points because of $\deg R(\lambda)=2N+2$. Under an alternative local coordinate $z=\lambda^{-1}$, they are viewed as $\infty_1=(0,1)$ and $\infty_2=(0,-1)$. We fix a set of regular cycle paths on $\Gamma$: $a_1,a_2,\cdots,a_N$; $b_1,b_2,\cdots,b_N$, which are independent and have intersection numbers as

$$a_i\circ a_j=b_i\circ b_j=0,\quad a_i\circ b_j=\delta_{ij},\qquad i,j=1,2,\cdots,N.$$

It is known that

$$\tilde\omega_l=\frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}},\qquad 1\le l\le N,$$

are $N$ linearly independent homomorphic differentials of $\Gamma$. Defining

$$A_{ij}=\int_{a_j}\tilde\omega_i,\qquad C=(A_{ij})^{-1},\qquad 1\le i,j\le N,\tag{4.15}$$

whence, we obtain a new normalized basis $\omega_j$

$$\omega_j=\sum_{l=1}^{N}C_{jl}\tilde\omega_l,\quad \int_{a_i}\omega_j=\sum_{l=1}^{N}C_{jl}\times\int_{a_i}\tilde\omega_l=\sum_{l=1}^{N}C_{jl}A_{li}=\delta_{ji},$$

and defining

$$B_{ij} = \int_{b_j} \omega_i, \qquad 1 \le i, j \le N,$$

a matrix that will be used later to construct the Riemann theta functions of $\Gamma$. Having a fixed point $p_0$, the Abel-Jacobi coordinates are defined as

$$\begin{aligned} \rho_j^{(1)}(x,y,t) &= \sum_{k=1}^{N} \int_{p_0}^{\mu_k(x,y,t)} \omega_j \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl} \int_{p_0}^{\mu_k} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \\ \rho_j^{(2)}(x,y,t) &= \sum_{k=1}^{N} \int_{p_0}^{\nu_k(x,y,t)} \omega_j \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl} \int_{p_0}^{\nu_k} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \end{aligned} \qquad 1 \le j \le N. \tag{4.16}$$

It is easy from $(4.16)_1$ to calculate that

$$\begin{aligned} \partial_x \rho_j^{(1)} &= \sum_{l=1}^{N}\sum_{k=1}^{N} C_{jl} \frac{\mu_k^{l-1}\mu_{k,x}}{\sqrt{R(\mu_k)}} \\ &= \sum_{l=1}^{N}\sum_{k=1}^{N} C_{jl} \frac{-\mu_k^{l-1}}{\prod\limits_{i=1, i\neq k}^{N} (\mu_k - \mu_i)}. \end{aligned} \tag{4.17}$$

Taking advantage of the following known formulae [2],

$$\begin{aligned} I_s &\triangleq \sum_{k=1}^{N} \frac{\mu_k^s}{\prod\limits_{i=1,i\neq k}^{N}(\mu_k - \mu_i)} = \delta_{s,N-1}, \quad 1 \le s \le N-1, \\ I_N &= \sigma_1 I_{N-1}, \qquad I_{N+1} = \sigma_1 I_N - \bar{\sigma}_1 I_{N-1}. \end{aligned} \tag{4.18}$$

Therefore,

$$\partial_x \rho_j^{(1)} = -\Omega_j^{(0)}, \qquad \Omega_j^{(0)} = C_{jN}, \qquad 1 \le j \le N. \tag{4.19}$$

Likewise, we can work out

$$\partial_y \rho_j^{(1)} = \Omega_j^{(1)}, \qquad \partial_t \rho_j^{(1)} = \Omega_j^{(2)}, \tag{4.20}$$

$$\partial_x \rho_j^{(2)} = \Omega_j^{(0)}, \qquad \partial_y \rho_j^{(2)} = -\Omega_j^{(1)}, \qquad \partial_t \rho_j^{(2)} = -\Omega_j^{(2)}, \tag{4.21}$$

where

$$\begin{aligned} \Omega_j^{(1)} &= C_{jN-1} + \sigma C_{jN}, \\ \Omega_j^{(2)} &= C_{jN-2} + \sigma C_{jN-1} + \sigma^2 C_{jN} - \bar{\sigma} C_{jN}. \end{aligned}$$

To summarize, it signifies that $\rho_j^{(1)}$ and $\rho_j^{(2)}$ can be regarded as linear superpositions,

$$\begin{aligned} \rho_j^{(1)} &= -\Omega_j^{(0)} x + \Omega_j^{(1)} y + \Omega_j^{(2)} t + \gamma_j^{(1)}, \\ \rho_j^{(2)} &= \Omega_j^{(0)} x - \Omega_j^{(1)} y - \Omega_j^{(2)} t + \gamma_j^{(2)}, \end{aligned} \qquad 1 \le j \le N, \tag{4.22}$$

with integral constants

$$\gamma_j^{(1)} = \sum_{k=1}^{N} \int_{p_0}^{\mu_k(0,0,0)} \omega_j, \qquad \gamma_j^{(2)} = \sum_{k=1}^{N} \int_{p_0}^{\nu_k(0,0,0)} \omega_j.$$

## 5 The quasi-periodic solutions

Seen in the above presentation, (4.22) provides the specially explicit solutions in the Abel-Jacobi coordinates $\rho^{(1)}$, $\rho^{(2)}$ of the KP and mKP equations. In order to obtain the expression of solutions in the original coordinate $g$, $h$ pertinent to the KP and mKP equations, we discuss the Jacobi inversion procedure,

$$\left(\rho^{(1)}, \rho^{(2)}\right) \Longrightarrow (\mu_k, \nu_k) \Longrightarrow (p, q) \Longrightarrow (g, h).$$

Let $T$ be the lattice in $\mathbb{C}^N$ generated by $2N$ vectors $\{\delta_i, B_j\}$. Then we obtain the Jacobian $T(\Gamma) = \mathbb{C}^N/T$. The Abel map $\mathcal{A}$

$$\mathcal{A}: \quad \mathrm{Div}(\Gamma) \to \mathrm{J(T)},$$

is defined setting

$$\mathcal{A}(p) = \left( \int_{p_0}^{p} \omega_1, \cdots, \int_{p_0}^{p} \omega_N \right),$$

where $p$ is an arbitrary point of $\Gamma$. Moreover, $\mathcal{A}$ can be linearly extended into arbitrary divisors $\mathcal{A}(\sum n_k p_k) = \sum n_k \mathcal{A}(p_k)$. From [28,29], the Riemann theta functions of $\Gamma$ are defined as

$$\begin{aligned} \theta(\zeta) &= \sum_{z\in\mathbb{Z}^N} \exp\left(\pi i \langle Bz, z\rangle + 2\pi i \langle \zeta, z\rangle\right), \quad \zeta \in \mathbb{C}^N, \\ (Bz, z) &= \textstyle\sum_{i,j=1}^{N} B_{ij} z_i z_j, \qquad (\zeta, z) = \sum_{i=1}^{N} z_i \zeta_i. \end{aligned}$$

Consider two special divisors $\sum_{k=1}^{N} p_k^{(m)}$,

$$\mathcal{A}\Big(\sum_{k=1}^{N} p_k^{(m)}\Big) = \sum_{k=1}^{N} \mathcal{A}\big(p_k^{(m)}\big) = \sum_{k=1}^{N} \int_{p_0}^{p_k^{(m)}} \omega = \rho^{(m)}, \qquad m = 1, 2,$$

where $p_k^{(1)} = (\mu_k, \zeta(\mu_k))$ and $p_k^{(2)} = (\nu_k, \zeta(\nu_k))$. In accordance with the Riemann theorem [28], there exist Riemann constants $M^{(1)}, M^{(2)} \in \mathbb{C}^N$ determined by $\Gamma$ itself such that

- $f^{(1)}(\lambda) \triangleq \theta(\mathcal{A}(\zeta(\lambda)) - \rho^{(1)} - M^{(1)})$ has exactly $N$ zeros at $\mu_1, \cdots, \mu_N$;
- $f^{(2)}(\lambda) \triangleq \theta(\mathcal{A}(\zeta(\lambda)) - \rho^{(2)} - M^{(2)})$ has exactly $N$ zeros at $\nu_1, \cdots, \nu_N$.

To make the functions single valued, the Riemann surface $\Gamma$ is cut along by all $a_k, b_k$ to form a simply connected region, whose boundary is denoted by $\gamma$. From the residue formula, we have

$$\begin{aligned} \sum_{j=1}^{N} \mu_j &= \frac{1}{2\pi i} \oint_\gamma \lambda d \ln f^{(1)}(\lambda) - \sum_{s=1}^{2} \operatorname*{Res}_{\lambda=\infty_s} \lambda d \ln f^{(1)}(\lambda), \\ \sum_{j=1}^{N} \nu_j &= \frac{1}{2\pi i} \oint_\gamma \lambda d \ln f^{(2)}(\lambda) - \sum_{s=1}^{2} \operatorname*{Res}_{\lambda=\infty_s} \lambda d \ln f^{(2)}(\lambda). \end{aligned} \tag{5.1}$$

Following [10], it is known that integrals

$$\frac{1}{2\pi i}\oint_{\gamma}\lambda d\ln f^{(m)}(\lambda)=\sum_{j=1}^{N}\int_{a_j}\lambda\omega_j \triangleq I(\Gamma), \qquad m=1,2,$$

are constants independent of $\rho^{(m)}$. The only remaining requirement is the calculation of residues,

$$\begin{aligned}
f^{(m)}(\lambda)|_{\lambda=\infty_s} &= \theta\left(\int_{p_0}^{p}\omega-\rho^{(m)}-M^{(m)}\right)\\
&=\theta\left(\int_{\infty_s}^{p}\omega-\pi_s-\rho^{(m)}-M^{(m)}\right)\\
&=\theta\left(\cdots,\int_{\infty_s}^{p}\omega_j-\pi_{sj}-\rho_j^{(m)}-M_j^{(m)},\cdots\right)\\
&=\theta\Big(\cdots,\rho_j^{(m)}+M_j^{(m)}+\pi_{sj}+(-1)^s\\
&\quad\times\left(C_{jN}z+\frac{1}{2}(C_{jN-1}+\sigma C_{jN})z^2+\cdots\right),\cdots\Big)\\
&=\theta_s^{(m)}(\rho^{(m)}+M^{(m)}+\pi_s)+(-1)^{s+m}\theta_{s,x}^{(m)}z+\cdots,
\end{aligned}$$

where

$$\pi_{sj}=\int_{\infty_s}^{p_0}\omega_j, \qquad s,m=1,2.$$

Consequently,

$$\operatorname*{Res}_{\lambda=\infty_s}\lambda d\ln f^{(m)}(\lambda)=(-1)^{s+m}\partial\ln\theta_s^{(m)}, \tag{5.2}$$

where

$$\begin{aligned}
\theta_s^{(1)}&=\theta\left(-\Omega^{(0)}x+\Omega^{(1)}y+\Omega^{(2)}t+\Upsilon_s\right),\\
\theta_s^{(2)}&=\theta\left(\Omega^{(0)}x-\Omega^{(1)}y-\Omega^{(2)}t+\Lambda_s\right),
\end{aligned}$$

with

$$\Upsilon_{sj}=\gamma_j^{(1)}+M_j^{(1)}+\pi_{sj}, \qquad \Lambda_{sj}=\gamma_j^{(2)}+M_j^{(2)}+\pi_{sj}, \quad 1\le j\le N.$$

It follows from (5.1) and (5.2) that

$$\sum_{l=1}^{N}\mu_l=I(\Gamma)+\partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}, \qquad \sum_{l=1}^{N}\nu_l=I(\Gamma)+\partial_x\ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}. \tag{5.3}$$

Substituting (5.3) into (4.4), we eventually derive a quasi-periodic solution of the mKP equation

$$g=-\partial_x\ln\frac{\theta(\Omega^{(0)}x-\Omega^{(1)}y-\Omega^{(2)}t+\Lambda_1)}{\theta(\Omega^{(0)}x-\Omega^{(1)}y-\Omega^{(2)}t+\Lambda_2)}-I(\Gamma)+\sigma, \tag{5.4}$$

and a special form of quasi-periodic solution of the KP equation

$$\begin{aligned}
h=&\frac{\theta\left(-\Omega^{(0)}x+\Omega^{(1)}y+\Omega^{(2)}t+\Upsilon_2\right)}{\theta\left(-\Omega^{(0)}x+\Omega^{(1)}y+\Omega^{(2)}t+\Upsilon_1\right)}\\
&\times\frac{\theta\left(\Omega^{(0)}x-\Omega^{(1)}y-\Omega^{(2)}t+\Lambda_2\right)}{\theta\left(\Omega^{(0)}x-\Omega^{(1)}y-\Omega^{(2)}t+\Lambda_1\right)}\\
&\frac{\theta\left(\Omega^{(1)}y+\Omega^{(2)}t+\Upsilon_1\right)}{\theta\left(\Omega^{(1)}y+\Omega^{(2)}t+\Upsilon_2\right)}\frac{\theta(-\Omega^{(1)}y-\Omega^{(2)}t+\Lambda_1)}{\theta(-\Omega^{(1)}y-\Omega^{(2)}t+\Lambda_2)}h(0,y,t),
\end{aligned} \tag{5.5}$$

which is different from the well-known expression $h(x,y,t)=2\partial_x^2\ln\theta(\Omega_1x+\Omega_2y+\Omega_3t+\Omega_0)+h_0$ in the references [5,13].

In conclusion, it is very difficult for a given (2+1)-dimensional nonlinear evolution equation to be decomposed into two (1+1)-dimensional soliton equations in the same hierarchy. Here we recover the (2+1)-dimensional integrable models of mKP and KP equations from two soliton equations in the cKdV hierarchy. The mKP and KP equations are conditionally decomposed into (1+1)-dimensional components that are easier to tackle with available tools. Along with this idea, soliton equations are further reduced into integrable FDHSs, which can be linearized on the Jacobian of a Riemann surface. Appealing to the Jacobi inversion, some special quasi-periodic solutions to the KP and mKP equations are simultaneously derived through a different decomposition to those previously given in the literature. Meanwhile, this shows that multi-dimensional nonlinear evolution equations possess abundant and various solutions from diverse constructions. Theoretically speaking, some other known (2+1)-dimensional nonlinear evolution equations can be similarly studied in this constructive framework. For this statement, we will provide some other examples in the near future.

This work was supported by the National Natural Science Foundation of China (No.10471132) and the Special Foundation for Chinese Major State Basic Research Project "Nonlinear Science". The authors are much obliged to the anonymous referees for their valuable remarks and suggestions.

## References

1. S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, *Theory of solitons, the inverse scattering methods* (New York, Consultants Bureau, 1984)
2. A.C. Newell, *Solitons in Mathematics and Physics* (PA, SIAM, Philadelphia, 1985)
3. M.J. Ablowitz, H. Segur, *Solitons and the inverse scattering transform* (PA, SIAM, Philadelphia, 1981)
4. V.B. Matveev, M.A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991)
5. E.D. Belokolos, A.I. Bobenko, V.Z. Enol'skii, A.R. Its, V.B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Springer, Berlin, 1994)

6. B.G. Konopilchenko, *Solitons in Multidimensions, Inverse Spectral Transform Method* (World Scientific, Singapore, 1993)
7. E. Zakharov, A.B. Shabat, Funct. Anal. Appl. **8**, 226 (1974)
8. C.W. Cao, Sci. China A **33**, 528 (1990)
9. C.W. Caoand, X.G. Geng, in *Proc. Conf. on Nonlinear Physics, Shanghai 1989, Research Reports in Physics* (Springer, Berlin, 1990), pp. 68-78
10. R.G. Zhou, J. Math. Phys. **38**, 2535 (1997)
11. X.G. Geng, Y.T. Wu, C.W. Cao, J. Phys. A: Math. Gen. **32**, 3733 (1999)
12. H.-H. Dai, X.G. Geng, Chaos, Solitons & Fractals **14**, 489 (2002)
13. C.W. Cao, Y.T. Wu, X.G. Geng, J. Math. Phys. **40**, 3948 (1999)
14. J.B. Chen, Chaos, Solitons & Fractals **19**, 905 (2004)
15. J.B. Chen, X.G. Geng, J. Phys. Soc. Jpn. **74**, 2217 (2005)
16. Y. Cheng, Y.S. Li, Phys. Lett. A, **157** 22 (1991)
17. Y. Cheng, Y.S. Li, J. Phys. A: Math. Gen. **25**, 419 (1992)
18. B.B. Kadometsev, V.I. Petviashvili, Sov. Phys.-Dokl. **15**, 539 (1970)
19. B.G. Konopelchenko, V.G. Dubrovsky, Phys. Lett. A **102** 15 (1984)
20. D. Levi, A. Sym, S. Wojciechowsk, J. Phys. A: Math. Gen. **16**, 2423 (1983)
21. C.W. Cao, X.G. Geng, J. Phys. A: Math. Gen. **23**, 4117 (1990)
22. Y.B. Zeng, Y.S. Li, J. Phys. A: Math. Gen. **26**, L273 (1993)
23. L.D. Faddeev, L.A. Takhtajian, *Hamiltonian methods in the theory of soliton* (Spring-Verlag, Berlin, 1987)
24. O. Babelon, C.M. Villet, Phys. Lett. B **237**, 411 (1990)
25. V.I. Arnold, *Mathematical methods of classical mechanics* (Springer, Berlin, 1978)
26. J. Avan, M. Talon, Phys. Lett. B **268**, 209 (1991)
27. O. Babelon, D. Bernard, M. Talon, *Introduction to classical integrable systems* (Cambridge University Press, 2003)
28. P. Griffiths, J. Harris, *Principles of algebraic geometry* (New York, Wiley, 1994)
29. D. Mumford, *Tata lectures on theta* (Birkhauser, Boston, 1984)